

Knot Embeddings in Improper Foldings

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Abstract

We demonstrate a connection between knot theory and origami folding which is permitted by an extended definition of folding that allows for the self-intersection of paper. In particular we prove that, under this extension, a folding is physically realizable if and only if all Jordan curves in the paper are mapped to the unknot under the folding operation. In cases where a folding is not physically realizable (and thus admits nontrivial knots), a number of questions arise about the relationship between the complexity of the folding map and the complexity of the knots produced.

Our connection with knot theory requires a definition of folding that allows for self-intersection yet retains information about the ordering of layers of paper which are pressed together. We formalize this in two steps. First, we define an *idealized folding* which captures folding with zero-thickness paper:

Definition 1. An *idealized folding* is a piecewise-differentiable arcwise-isometric mapping $(0, 1)^2 \rightarrow \mathbb{R}^3$.

Idealized foldings are natural models for flat-folded origami, where all faces lie in the same plane. This definition allows for self-intersection but does not necessarily define a layer ordering, so we next define *realizations* of an idealized folding which do carry layer ordering information, distinguishing between two types: *proper* realizations which correspond to physically possible foldings, and *improper* realizations which involve self-intersection. This distinction corresponds to the (non-)existence of injective maps which are “close-enough” to an idealized folding.

Definition 2. Suppose f is an idealized folding. Then a *realization* of f is a homotopy $F : (0, 1)^2 \times [0, 1] \rightarrow \mathbb{R}^3$ such that

1. For all $0 \leq \varepsilon \leq 1$, $x \in (0, 1)^2$ we have $d(F(x, 0), F(x, \varepsilon)) \leq \varepsilon$ where $d(\cdot, \cdot)$ is the Euclidean metric (in particular, $F(\cdot, 0) = f$),
2. For $0 < \varepsilon \leq 1$, all self-intersections in $F(\cdot, \varepsilon)$ are transversal.

If a realization is injective for all $0 < \varepsilon \leq 1$ we call it *proper*; otherwise we say it is *improper*.

Real paper has nonzero thickness and we contend the above definition captures this fact: as ε goes to 0, this homotopy may be interpreted as a ‘thinning’ of a physical realization of the

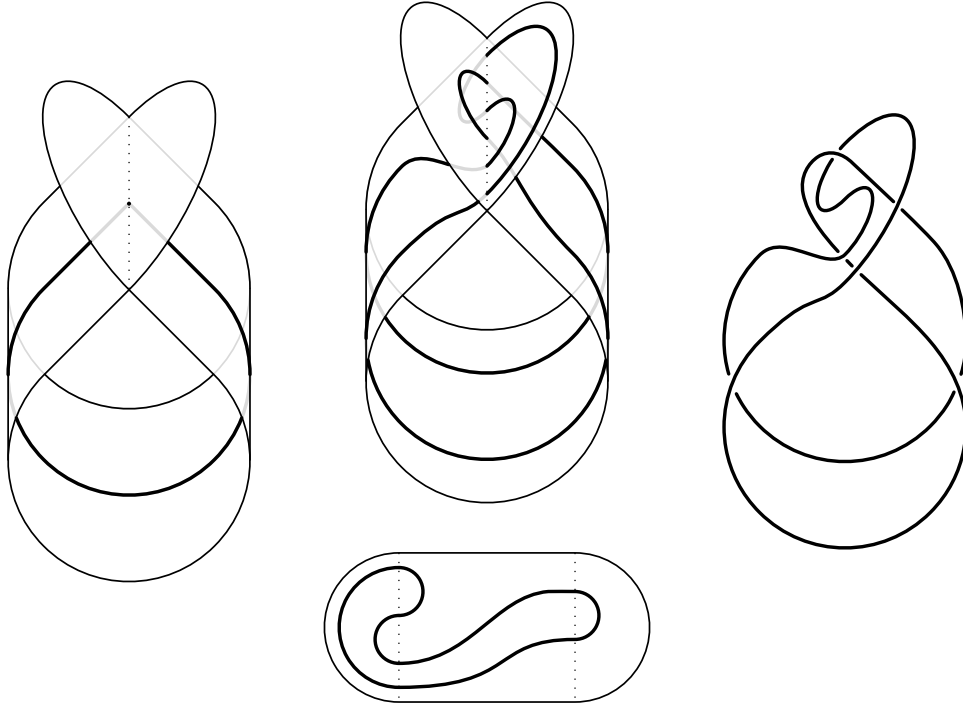


Figure 1: (Left) Locating the embedding region. The bold line is the path p and the dashed line is the path with two preimages; (Middle) The knot embedding shown in bold and its preimage; (Right) The resulting knot.

folding down to a limit defined by the idealized folding. We are now prepared to introduce knots to foldings.

Definition 3. A *knot embedding* in a realization F is any Jordan curve g in $(0, 1)^2$ along with a ε , $0 < \varepsilon \leq 1$ such that $F(g, \varepsilon)$ is homeomorphic to the circle.

Note that not all g are knot embeddings; it may be that $F(g, \varepsilon)$ includes self-intersection and is thus not homeomorphic to the circle.

Theorem 1. A realization is proper if and only if for all knot embeddings (g, ε) , the image $F(g, \varepsilon)$ is the unknot.

Proof sketch. The “only if” is clear. We prove the “if” direction via the contrapositive. If for some ε there is self-intersection in the realization then there is a path in the image of the paper which has two distinct preimages in the domain. The paper is connected, so we may find a “nice” path p from the midpoint of one preimage to the other. Take a δ -neighborhood about p . We may now embed a nontrivial knot in this neighborhood as sketched in Figure 1. \square

We are left with a number of open questions of which we make initial explorations in the full paper, including:

- For a given knot, how many creases are required for an idealized folding to admit an embedding for that knot? Does this lead to a useful knot invariant?
- What classes of knots may be embedded with a single region of self-intersection, as pictured in Figure 1? With n regions?